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Painlevé equations as classical analogues of Heun equations

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Abstract. The relationship between the Heun class of second-order linear equations and the Painlevé second-order nonlinear equations is studied. The symbol of the Heun class equations is regarded as a quantum Hamiltonian. The independent variable and the differentiation operator correspond to the canonical variables and one of the parameters of the equation is assumed to be time. Painlevé equations appear to be Euler–Lagrange equations related to corresponding classical motion.

1. Introduction

Two physical theories can be related to a given Hamiltonian $H(p, q, t)$ determined as a function of the canonical variables q, p and time t. It could either be classical mechanics or it could be quantum theory. In the first case the classical motion is described by the functions $q(t)$, $p(t)$—solutions of the Hamilton system of equations. In the Lagrange formalism one can study only the function $q(t)$, regarding it as a solution of the Euler–Lagrange second-order ordinary differential equation (ODE).

In the second case q, p are treated as operators $\hat{q}, \hat{p}$ acting on the wavefunction $\psi(x, t)$ which is a solution of the corresponding Schrödinger equation. Classical objects $\bar{q}(t)$, $\bar{p}(t)$ called observables are calculated as matrix elements of $\hat{p}$, $\hat{q}$ with the given $\psi(x, t)$. Transforms from a quantum problem to the related classical problem and *vice versa* have been widely studied for different concrete Hamiltonians and, consequently, different physical systems.

Equations with solutions having sufficiently simple characteristics as functions of complex independent variables are distinguished from the others. Taken as Euler–Lagrange equations related to appropriate Hamiltonians, the so-called Painlevé equations enjoy the Painlevé property. This means that movable singularities of all solutions of these equations are poles (no movable branching points or essential singularities). Specially chosen solutions of the Painlevé equations constitute the class of special functions related to nonlinear mathematical physics.

On the other hand, the one-dimensional Schrödinger equations can be classified according to the number and the rank of singularities. The simplest equations constitute the so-called hypergeometric class of differential equations. Next in complexity comes the Heun class which arises from the Heun equation characterized by four regular singularities. Specially chosen solutions of equations belonging either to the hypergeometric class or the

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Heun class constitute the class of special functions related to linear mathematical physics [4].

The object of this paper is to describe the relationship between the equations belonging to the Heun class and the corresponding Painlevé equations. This relationship is based on the existence of common Hamiltonians generating both equations. In the first case it is regarded as the quantum Hamiltonian and in the second case as the classical one. In this sense, Painlevé equations may be called classical analogues of Heun's equations.

It is already known that every Painlevé equation (nonlinear) is related to linear equations. Namely, it can be obtained as a compatibility condition of two linear equations [5, 6]. The first one is the second-order linear equation with singularities, among which one is an apparent singularity. In the absence of the apparent singularity, this equation pertains to the Heun class but the presence of the singularity it pertains to the class next in complexity. The second linear equation is an equation preserving the monodromie of solutions of the first equation under variations of an appropriate parameter. This fact lays on the basis of Riemann–Hilbert theory and the so-called method of isomonodromic deformations. The modern presentation of the problem uses the first-order 2×2 system, called the Schlesinger system [7], instead of the original second-order equation.

Our aim is to construct a straightforward relationship between more simple linear second-order equations (namely of Heun's class) and Painlevé equations beyond the scope of the method of isomonodromic deformation purely on the basis of comparison between quantum and classical dynamics.

We hope that the absence of the Planck constant and imaginary unity in the presentation of operator $\hat{p}$ will not lead to any misunderstanding.

2. Basic definitions

The Heun class of differential equations comprises the Heun equation (second-order linear homogeneous ODE with four regular singularities) and all equations generated from it by specialization of parameters or by confluence processes. The presentation of the processes of confluence for the Fuchsian equations and for the Heun equation in particular has been described in [1–3]. The process of confluence is controlled by numerical characteristic of the singularities—their s-rank. The s-rank of a regular singularity is equal to unity. The s-rank is an irregular singularity may be either integer or half-integer. In the first case we call the singularity unramified. In the second case the singularity is ramified. Equations with ramified irregular singularities are called ramified. Formula for the evaluation of the s-rank and the theorem of subadditivity of the s-rank at confluence are given in [3].

The basic Heun equation in the canonical natural form [2] reads

$$
\begin{aligned}
&L_z^{(1,1,1;1)}(a,b,c,d;t;\sigma)y(z) \\
&\qquad = \left[\mathrm{D}^2 + \left(\frac{c}{z} + \frac{d}{z-1} + \frac{e}{z-t}\right)\mathrm{D} + \left(\frac{ab(z-t)-\sigma}{z(z-1)(z-t)}\right)\right]y(z) = 0 \\
&\qquad D = \frac{\mathrm{d}}{\mathrm{d}z}.
\end{aligned} \tag{1}
$$

Here $1-c$, $1-d$, $1-e$ are characteristic exponents of Frobenius-type solutions at the points $z = 0$, $z = 1$, $z = \infty$, respectively (others are zero) and a, b are corresponding characteristic exponents at infinity. Since one of the characteristic exponents at all finite singularities is taken to be equal to zero equation (1) is stated as being in canonical form [2].

The characteristic exponents satisfy the Fuchs condition

$$a + b + 1 = c + d + e. \tag{2}$$

The set of s-ranks of the singularities is indicated as the upper index for the differential operator L. The same indication will used throughout this paper.

Parameter σ does not influence the major local characteristics of solutions of equation (1) and is called the accessory parameter.

Equation (1) can be rewritten in terms declared in (12) as

$$Hy = \frac{1}{f(t)}[r_3(q,t)p^2 + r_2(q,t)p + r_1(q,t)]y = \lambda y \tag{3}$$

with

$$\begin{aligned} &r_1(q,t) = ab(q-t) \qquad r_2(q,t) = c(q-1)(q-t) + dq(q-t) + eq(q-1) \\ &r_3(q,t) = q(q-1)(q-t) \qquad \sigma = \lambda t(t-1) \qquad f(t) = t(t-1) \\ &1 - c = \theta_1 \qquad 1 - d = \theta_2 \qquad 1 - e = \theta_3 \qquad b - a = \theta_4 \end{aligned} \tag{4}$$

and z, D substituted for q, p. It is necessary to stress that parameters θ_j are differences of characteristic exponents at singularities and therefore they are invariants of s-homotopic transformations of the dependent variable and Mœbius transformation of the independent variable. The normalization of parameter λ is predicted by the fact that

$$\lambda = \mathrm{res}_{q=t}\frac{\sigma - r_1}{r_3} \tag{5}$$

so that the location of the singular point and the residue of the 'potential' at this point are chosen to be canonically adjoint variables related to 'time' and 'energy'.

By Painlevé equations we mean basically six equations which are denote as

$$P^{(6)} \qquad P^{(5)} \qquad P^{(4)} \qquad P^{(3)} \qquad P^{(2)} \qquad P^{(1)}$$

respectively.

Standard Painlevé equations in notation (13) are the following [7]:

$$\begin{aligned} P^{(6)}: \quad q_{tt} &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} - \frac{1}{q-t}\right)q_t^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)q_t \\ &\quad + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right) \end{aligned} \tag{6}$$

$$P^{(5)}: \quad q_{tt} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)q_t^2 - \frac{1}{t}q_t + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) - \frac{\gamma q}{t} - \frac{\delta q(q+1)}{q-1} \tag{7}$$

$$P^{(4)}: \quad q_{tt} = \frac{1}{2q}q_t^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - \alpha)q + \frac{\beta}{q} \tag{8}$$

$$P^{(3)}: \quad q_{tt} = \frac{1}{q}q_t^2 - \frac{1}{t}q_t + \frac{1}{t}(\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q} \tag{9}$$

$$P^{(2)}: \quad q_{tt} = q^3 + tq + \alpha \tag{10}$$

$$P^{(1)}: \quad q_{tt} = 6q^2 + t. \tag{11}$$

It is necessary to stress that 'in general' equation (7) contains three arbitrary parameters and equation (8) two parameters. Under scaling transformations it is always possible to fix one parameter in (7) and two parameters in (8) unless they are not zeros. The latter case will be discussed separately below.

The starting linear 'quantum' equation (namely, of the Heun's class in our consideration) is taken in the form

$$H(q, p, t, \theta_j)y(q, t) = \lambda y(q, t). \tag{12}$$

Here q is the independent complex variable, p is the differentiation operator over q, θ_i are 'local' parameters exposing local behaviour of the solutions near the singularities, t is a 'scaling' parameter and λ is a 'global' parameter which usually plays the role of spectral parameter and in the theory of Heun equations is called the accessory parameter [2]. The function H in (12) is supposed to be the quantum Hamiltonian.

For a given equation of Heun type, presentation (12) is by no means unique. We can always redefine the spectral parameter by multiplication of both sides of (12) by an arbitrary function $g(t, \theta_j)$ or by addition of any function $g_1(t, \theta_j)$ to both sides of (12). Moreover, equations (12) are usually studied under an equivalence relation originated by s-homotopic transformations of dependent variable y [2, 3] and isomorphisms of the complex plane q.

For the same Hamiltonian (12), studied as a classical object, one can write Euler–Lagrange equations of classical motion of the form

$$q_{tt} = F(q, q_t, t, \theta_i). \tag{13}$$

The main goal of this paper is to prove the following theorem.

Theorem 1. Every Painlevé equation can be obtained as an Euler–Lagrange equation of the form (13) generated by the Hamiltonian related to the appropriate linear second-order equation (12). The latter one belongs to the Heun class.

3. Proof of theorem 1

The first (equation (6)) of the listed Painlevé equations in the frame of comparison between (12) and (13) is generated by the basic Heun equation (1). The corresponding Lagrangian reads

$$L = \frac{f}{4r_3}\left(q_t - \frac{r_2}{f}\right)^2 - \frac{r_1}{f}. \tag{14}$$

The following Euler–Lagrange equation results:

$$q_{tt} = \frac{1}{2}\frac{\partial}{\partial q}(\ln r_3)q_t^2 - \left(\frac{\partial}{\partial t}(\ln f) - \frac{\partial}{\partial t}(\ln r_3)\right)q_t + \frac{r_3}{f^2}\left(\frac{\partial}{\partial q}\frac{r_2^2}{2r_3} + f\frac{\partial}{\partial t}\frac{r_2}{r_3} - 2\frac{\partial r_1}{\partial q}\right). \tag{15}$$

This coincides with equation (6) with the following correspondence of local parameters:

$$\alpha = \frac{(c+d+e)^2}{2} - 2ab \qquad \beta = -\frac{c^2}{2} \qquad \gamma = \frac{d^2}{2} \qquad \delta = -\frac{(1-e)^2 - 1}{2}. \tag{16}$$

Next comes $P^{(5)}$ generated by the confluent Heun equation. The following comment is needed in advance. There exist two modifications of the confluent Heun equation which differ by the s-rank $R(z^*)$ of the irregular singularity z^*. This either holds for $R = 2$ and normal formal asymptotic solutions can be constructed in the vicinity of this point, or it holds for $R = \frac{3}{2}$ and subnormal formal asymptotic solutions can be constructed in the vicinity of this point. In the case of the first modification the process of confluence from (9) is arranged by the following limiting transform,

$$t \mapsto 1 + \epsilon t \qquad d \mapsto \frac{1}{\epsilon} + d \qquad e \mapsto \frac{1}{\epsilon} \qquad \epsilon \to 0 \tag{17}$$

which leads to coalescence of the regular singularities $z = t$ and $z = 1$ into an irregular singularity $z = 1$. The resulting equation reads

$$L_z^{(1,1,2)}(c, d; a, t; \sigma)y(z) = (z(z-1)^2\mathrm{D}^2 + (tz + c(z-1)^2 + dz(z-1))\mathrm{D} + (ab(z-1) - \sigma))y(z) = 0. \tag{18}$$

The corresponding Hamiltonian in form (3) is determined by

$$r_1(q,t) = ab(q-1) \qquad r_2(q,t) = c(q-1)^2 + dq(q-1) - tq$$
$$r_3(q,t) = q(q-1)^2 \qquad \sigma = \lambda t \qquad f(t) = t. \tag{19}$$

The Euler–Lagrange equation related to this Hamiltonian coincides with (1) with the following relationship between the parameters

$$\alpha = \frac{(c+d)^2}{2} - 2ab \qquad \beta = -\frac{c^2}{2} \qquad \gamma = \frac{d^2}{2} \qquad \delta = -1. \tag{20}$$

If it holds $\delta \neq 0$, then by scaling the dependent variable in (7) this parameter can be always made equal to unity. Hence we get what is needed. In order to study the case $\delta = 0$ the ramified confluent Heun equation (RCHE) with irregular singularity at $z = 1$ characterized by s-rank $R = \frac{3}{2}$ can be chosen as a starting equation:

$$L_z^{(1,1;3/2)}(c, d; t; \sigma)y(z) = (z(z-1)^2\mathrm{D}^2 - (c(z-1) + dz(z-1))\mathrm{D} + (-tz/(z-1) - \sigma))y(z) = 0. \tag{21}$$

The corresponding Euler–Lagrange equation is $P^{(5)}$ with

$$\alpha = \frac{d^2}{2} \qquad \beta = -\frac{c^2}{2} \qquad \gamma = 2 \qquad \delta = 0. \tag{22}$$

Further scaling transformations of the dependent variable can make parameter γ arbitrary.

If three regular singularities in Heun equation (1) coalesce at infinity the resulting equation is called the biconfluent Heun equation (BHE). In the canonical form it reads

$$L_z^{(1;3)}(c; a, t, \sigma)y(z) = (z\mathrm{D}^2 + (-z^2 - tz + c)\mathrm{D} + (-az - \sigma))y(z) = 0. \tag{23}$$

The Hamiltonian is determined by

$$r_3(q,t) = q \qquad r_2(q,t) = c - tq - q^2 \qquad r_1(q,t) = -aq$$
$$\sigma = \lambda \qquad f(t) = 1 \tag{24}$$

and leads to the following Euler–Lagrange equation:

$$q_{tt} = \frac{q_t^2}{2q} + \frac{3}{2}q^3 + 2tq^2 + \left(\frac{t^2}{2} - c + 2a\right)q - \frac{c^2}{2q}. \tag{25}$$

Equation (25) under scaling of q and t can be transformed to $P^{(4)}$ in form (8).

The triconfluent Heun equation (THE) has only one irregular singularity which lays at infinity. THE is originated by a confluence process from equation (1). It reads

$$L_z^{(;4)}(a, t; \lambda)y(z) = (\mathrm{D}^2 + (-z^2 - t)\mathrm{D} + (-az - \lambda))y(z) = 0. \tag{26}$$

Under the procedure defined above one obtains a Euler–Lagrange equation in the form

$$q_{tt} = 2q^3 + 2tq + 2a \tag{27}$$

which is equivalent to $P^{(2)}$ after appropriate scaling. The Painlevé equation $P^{(1)}$ origins from the ramified triconfluent Heun equation

$$L_z^{(7/2)}(t; \lambda)y(z) = (\mathrm{D}^2 + (-z^3 - tz - \lambda))y(z) = 0 \tag{28}$$

where the singularity at infinity is characterized not by the s-rank $R = 4$ as in (26) but by the s-rank $R = \frac{7}{2}$. The corresponding Euler–Lagrange equation is

$$q_{tt} = 6q^2 + t. \tag{29}$$

The equation that gives some problems is $P^{(3)}$. First we transform $P^{(3)}$ to $\tilde{P}^{(3)}$

$$\tilde{P}^{(3)}: \qquad q_{tt} = \frac{1}{q}q_t^2 - \frac{1}{t}q_t + \frac{1}{t^2}(\alpha q^2 + \gamma q^3) + \frac{\beta}{t} + \frac{\delta}{q} \tag{30}$$

with the help of the substitution

$$t \mapsto \sqrt{t} \qquad q \mapsto \frac{q}{\sqrt{t}}. \tag{31}$$

Equation $\tilde{P}^{(3)}$ fits more to the confluence process and, of course, enjoys the Painlevé property. The double-confluent Heun equation (DHE) origins from the Heun equation (1) when two regular singularities coalesce at zero and two others coalesce at infinity. It reads

$$L_z^{(2;2)}(a, c, t; \sigma)y(z) = (z^2\mathrm{D}^2 + (-z^2 + cz - t)\mathrm{D} + (-az - \sigma))y(z) = 0. \tag{32}$$

The corresponding Hamiltonian

$$H = \frac{1}{t}(q^2p^2 + (-q^2 + cq + t)p - aq) \tag{33}$$

leads to the Euler–Lagrange equation of the form

$$q_{tt} = \frac{1}{q}q_t^2 - \frac{1}{t}q_t + \frac{q^2}{t^2}(2a - c) - \frac{c+1}{t} + \frac{1}{q} + \frac{q^3}{t^2}. \tag{34}$$

The case of zero coefficients, $P^{(3)}$ is treated once again by turning to the ramified equations with half-integer s-ranks. One of them can be written in the form

$$L_z^{(3/2;2)}(a, t; \sigma)y(z) = (z^2\mathrm{D}^2 - z^2\mathrm{D} + (-az - \sigma - t/z))y(z) = 0 \tag{35}$$

resulting in $P^{(3)}$ with

$$\alpha = 2a \qquad \beta = -2 \qquad \gamma = 1 \qquad \delta = 0. \tag{36}$$

The final possibility is to take an equation

$$L_z^{(3/2;3/2)}(t; \sigma)y(z) = (z^2\mathrm{D}^2 + (-z + \sigma - t/z))y(z) = 0 \tag{37}$$

where two singularities are characterized by the s-rank $R = \frac{3}{2}$ in order to satisfy $\gamma = 0$. Scaling transformations afford an arbitrariness of the two parameters that remain. This completes the proof of theorem 1. □

4. Discussion

From the point of view of the Heun class, the arrangement of Poincare equations seems to be not sufficiently satisfactory. In order to achieve better agreement between two classes of equations it is necessary to:

(i) interchange the singularities of $P^{(5)}$ at $q = 1$ and $q = \infty$;

(ii) substitute $P^{(3)}$ for $\tilde{P}^{(3)}$;

(iii) exclude $P^{(1)}$ as a separate equation and study it as a special reduction of $P^{(2)}$.

Moreover, there is no Painlevé equation related to the ramified biconfluent Heun equation with the s-rank of the singularity at infinity $R = \frac{5}{2}$. This prevents formulation of the theorem which would be inverse to theorem 1. It also presents the possibility that the list of Painlevé equations is not complete.

A further problem that could be studied is how the matrix elements—observables—for solutions of the Heun equation are related to solutions of Painlevé equations. A solution of this problem can give new approaches to the investigation of the asymptotics of nonlinear equations.

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